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Isomorphisms between quantum group covariant q-oscillator systems defined for q and q^{-1}

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Abstract. It is shown that there exists an isomorphism between q-oscillator systems covariant under $SU_q(n)$ and $SU_{q^{-1}}(n)$. By the isomorphism, the defining relations of a $SU_{q^{-1}}(n)$ covariant q-oscillator system are transmuted into those of $SU_q(n)$. It is also shown that a similar isomorphism exists for the system of q-oscillators covariant under the quantum supergroup $SU_q(n/m)$. Furthermore, the cases of q-deformed Lie (super)algebras constructed from covariant q-oscillator systems are considered. The isomorphisms between q-deformed Lie (super)algebras cannot be obtained by the direct generalization of the isomorphism for covariant q-oscillator systems.

1. Introduction

Since the discovery of quantum deformation (the so-called q-deformation) of Lie groups and Lie algebras [1–5], many q-deformed objects have been introduced. For example we can mention the q-deformed hyperplane [6], differential forms and derivatives on the q-deformed hyperplane [7], q-(super)oscillators [8, 9], q-deformed covariant oscillator systems [10–13] and their generalization [14], q-symplecton [12, 15], reflection equation algebras [16], and so on. Almost all of these objects are essentially defined by the same algebraic structure, that is, Zamolodchikov–Faddeev algebra [17] or quantum-group tensor [18]. Needless to say, tensors are fundamental quantities in physics. If q-deformed theories describe nature, then quantum-group tensors can also be fundamental quantities. If so, we will have one, or more, fundamental physical constants, i.e. deformation parameter(s). q-deformed objects may be regarded as 'functions' of deformation parameters. However, the relationship between q-deformed objects defined for different values of the deformation parameter q is unclear. This could be an important problem when physical applications of quantum-group tensors are considered.

This problem has been discussed for the q-oscillator $H_q = \{a, a^\dagger, N\}$ in two publications [19, 20]. Both authors found that the central element of H_q plays a crucial role when we consider two q-oscillator algebras defined for different values of the deformation parameter. In [19], Chaichian *et al* derived the formula which transforms the elements of H_q to the corresponding elements of $H_{q'}$ based on the assumption that the element N and the central element are independent of q. Without such an assumption, in this paper we discuss the special case of the problem, that is, the relationship between H_q and $H_{q^{-1}}$. Chaichian *et al* found the one-to-one correspondence between the elements of H_q and $H_{q^{-1}}$ which transmute the defining relations of $H_{q^{-1}}$ into those of H_q [20]. The elements of $H_{q^{-1}}$ can be expressed in terms of those of H_q; therefore, we can say that the algebra H_q is invariant under the replacement $q \leftrightarrow q^{-1}$. In mathematical language, H_q is isomorphic to $H_{q^{-1}}$.

In this paper we discuss the relationship between two q-deformed algebras defined for q and q^{-1}. This is the simplest case of the general problem of the relationship between q-deformed algebras defined for different values of deformation parameters. Furthermore, a Drinfeld–Jimbo deformation of Lie algebra is believed to be invariant under the replacement $q \leftrightarrow q^{-1}$. For example, $\{J_\pm, J_0\}$ and $\{\bar{J}_\pm, \bar{J}_0\}$ denote the generators of $U_q(sl(2))$ and $U_{q^{-1}}(sl(2))$, respectively, then $J_\alpha = \bar{J}_\alpha$, $\alpha = \pm, 0$. This result holds even when the $U_q(sl(2))$ is embedded into the direct product of two q-oscillators, $H_q \otimes H_q$ [20]. Such explicit relations are not yet known either for the quantum group $SU_q(2) \equiv Func_q(SU(2))$, which is dual to $U_q(sl(2))$, or for any quantum-group tensors. Such explicit relations may provide a new symmetry of the q-deformed theories under the replacement of $q \leftrightarrow q^{-1}$. As for the representation theory, our consideration will be a hint on how to construct new representations of q-deformed Lie algebras and is mentioned in the final part of the next section.

The q-deformed algebras discussed in this article are: (i) q-oscillator algebra which is covariant under the action of the quantum group $SU_q(n)$, (ii) q-oscillator algebra which is covariant under the quantum super group $SU_q(n/m)$, (iii) q-deformation of the Lie algebra $u(n)$ constructed from (i), and (iv) q-deformation of the Lie super algebra $u(n/m)$ constructed from (ii). The algebras (iii) and (iv) are also covariant under the action of the $SU_q(n)$ and $SU_q(n/m)$, respectively. The latter two cases are applications of the results obtained in the former two cases, since it is natural to consider the embedding of deformed Lie algebras into the q-deformed oscillator algebras.

This paper is organized as follows. In the next section, we briefly review our previous result on the q-oscillator algebra H_q. In section 3 it will be shown that, between $SU_q(n)$ and $SU_{q^{-1}}$ covariant q-oscillator systems, there exists the same isomorphism as for the case of H_q. By the isomorphism, the defining relations of the $SU_{q^{-1}}(n)$ covariant q-oscillator system are transmuted into those of $SU_q(n)$. A q-deformation of $u(n)$ will be constructed from the $SU_q(n)$ covariant q-oscillator system in section 4. The obtained q-deformed algebra is differerent from the well known Drinfeld–Jimbo deformation of $u(n)$. Unfortunately, the isomorphisms between covariant q-oscillator systems are not applicable in establishing a similar isomorphism between q-deformed Lie algebra. In section 5 similar results will be obtained for $SU_q(n/m)$ covariant q-oscillator system and q-deformed Lie super algebra constructed from them. Section 6 is devoted to a discussion.

2. Brief review of Biedenharn's q-oscillator

Before discussing q-oscillators which are covariant under quantum groups, let us briefly consider the case of the q-oscillator H_q introduced by Biedenharn [8]. The algebra H_q is generated by three elements $a, a^\dagger, N$ and they satisfy the relations

$$[N, a^\dagger] = a^\dagger \qquad [N, a] = -a \qquad aa^\dagger - qa^\dagger a = q^{-N}. \tag{2.1}$$

A central element of this algebra is given by

$$C = q^{-N}([N] - a^\dagger a) \tag{2.2}$$

where $[N] = (q^N - q^{-N})/(q - q^{-1})$. This algebra is defined for the deformation parameter q. We also define the q-oscillator defined for q^{-1}; it is denoted by $H_{q^{-1}}$ and its elements are denoted by $\bar{a}, \bar{a}^\dagger, \bar{N}$. The elements of $H_{q^{-1}}$ satisfy the same relation as (2.1) provided that q is replaced with q^{-1} in (2.1). It is possible to relate $H_{q^{-1}}$ to H_q in the following way [20].

Proposition 2.1. There exists an isomorphism $\varphi : H_q \to H_{q^{-1}}$ such that φ transmutes the defining relations of $H_{q^{-1}}$ into those of H_q. The explicit formulae are given by

$$\bar{N} = N \qquad \bar{a} = \frac{1}{\sqrt{F}}a \qquad \bar{a}^\dagger = \frac{1}{\sqrt{F}}a^\dagger \tag{2.3}$$

where F is a central element of H_q defined by

$$F = 1 - (q - q^{-1})C. \tag{2.4}$$

Proof. It is easy to verify, by direct calculation, that the defining relations of $H_{q^{-1}}$ are reduced to those of H_q by substituting relations (2.3) into the defining relations of $H_{q^{-1}}$. □

As is widely known, n copies of H_q can realize the q-deformed $u(n)$ introduced by Drinfeld and Jimbo. We can also realize the q-deformed $u(n)$ defined for q^{-1} by making use of $H_{q^{-1}}$. It should be emphasized that the realization in [8] must be modified when the central element (2.2) does not vanish. It has been shown [20] that both realizations are identical because of (2.3), namely, two q-deformed $u(n)$'s defined for q and q^{-1} are identical. The modified realization can give a new representation of the q-deformed $u(n)$ in which the elements of representation matrices diverge in the limit of $q \to 1$ [21].

3. $SU_q(n)$ covariant q-oscillator system

In this section a q-oscillator which is covariant under the action of the quantum group $SU_q(n)$ is considered. In order to distinguish it from the n copies of H_q, we call it the $SU_q(n)$ covariant q-oscillator system and denote it by $\mathcal{A}_q$. The algebra $\mathcal{A}_q$ is generated by $2n$ generators $\{A_i, A_i^\dagger,\ i = 1, \ldots, n\}$ and they satisfy the defining relations [10–12]

$$\begin{aligned} &A_i A_j = q A_j A_i \qquad A_i^\dagger A_j^\dagger = q^{-1} A_j^\dagger A_i^\dagger \qquad i < j \\ &A_i A_j^\dagger = q A_j^\dagger A_i \qquad i \neq j \\ &A_i A_i^\dagger - q^2 A_i^\dagger A_i = 1 + (q^2 - 1) \sum_{k=1}^{i-1} A_k^\dagger A_k. \end{aligned} \tag{3.1}$$

We assume $q \in \mathbb{R},\ q > 1$ hereafter. The *-anti-involution is introduced by

$$(A_i)^* = A_i^\dagger \qquad (A_i^\dagger)^* = A_i. \tag{3.2}$$

The q-annihilation operators and the q-creation operators are covariant and contravariant tensors of rank one under the co-action of $SU_q(n)$, respectively. This means that the transformations

$$A_i \to A_i' = \sum_{k=1}^{n} t_{ij} A_j \qquad A_i^\dagger \to A_i^{\dagger\prime} = \sum_{k=1}^{n} t_{ij}^* A_j^\dagger \tag{3.3}$$

preserve the defining relations of $\mathcal{A}_q$ (3.1). Here $t_{ij} \in SU_q(n)$ and we assume that t_{ij} commute with all the generators of $\mathcal{A}_q$. The commutation relations of t_{ij} are written using the R-matrix

$$RT \otimes T = (1 \otimes T)(T \otimes 1)R \tag{3.4}$$

where $T = (t_{ij})$ and the R-matrix is given by

$$R = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i > j} e_{ij} \otimes e_{ji} \tag{3.5}$$

where e_{ij} is the $n \times n$ matrix with entry 1 at position (i, j) and 0 elsewhere. The *-anti-involution is defined by

$$T^* = (t_{ji}^*) \qquad T^*T = TT^* = I$$

where I is the $n \times n$ unit matrix. The quantum determinant, which is the central element of $GL_q(n)$, is defined by $\det_q T = \sum_\sigma (-q)^{l(\sigma)} t_{1\sigma(1)} \cdots t_{n\sigma(n)}$, where $l(\sigma)$ is the minimal number of inversions in the permutation σ. We set $\det_q T = 1$.

We also define the $SU_{q^{-1}}(n)$ covariant q-oscillator system $\mathcal{A}_{q^{-1}}$ by replacing q with q^{-1} in (3.1); they form rank one tensors under the action of $SU_{q^{-1}}(n)$. We denote q-deformed objects defined for q^{-1} by attaching a bar on their elements, e.g. $\mathcal{A}_{q^{-1}} = \{\bar{A}_i, \bar{A}^\dagger{}_i\}$. Our aim is to establish a relationship between $\mathcal{A}_q$ and $\mathcal{A}_{q^{-1}}$. It should be noted that the trivial relation $\bar{A}_i = A_i$, $\bar{A}^\dagger{}_i = A_i^\dagger$ is prevented, since it concludes unacceptable results; $A_i A_j = 0$ etc.

It is possible to relate the elements of $\mathcal{A}_q$ and $\mathcal{A}_{q^{-1}}$ so that the covariant q-oscillator system is invariant under $q \leftrightarrow q^{-1}$.

Proposition 3.1. There exists an isomorphism $\varphi : \mathcal{A}_q \to \mathcal{A}_{q^{-1}}$ such that φ transmutes the defining relations of $\mathcal{A}_{q^{-1}}$ into those of $\mathcal{A}_q$. The explicit formulae are given by

$$\bar{A}_i = \Gamma_{i-1}^{-1}\Gamma_i^{-1} A_i \qquad \bar{A}_i^\dagger = A_i^\dagger \Gamma_{i-1}^{-1}\Gamma_i^{-1} \tag{3.6}$$

where

$$\Gamma_i \equiv \sqrt{[A_i, A_i^\dagger]} \qquad \Gamma_0 \equiv 1. \tag{3.7}$$

Proof. Using the properties of Γ_i we can prove the statement by direct calculations. Note that, using (3.1), Γ_i $(i \neq 0)$ is rewritten as

$$\begin{aligned}\Gamma_i &= \sqrt{A_{i+1}A_{i+1}^\dagger - q^2 A_{i+1}^\dagger A_{i+1}} \\ &= \sqrt{1 + (q^2-1)\sum_{k=1}^{i} A_k^\dagger A_k}.\end{aligned} \tag{3.8}$$

Γ_i is not affected by the *-anti-involution: $\Gamma_i^* = \Gamma_i$. From these facts we obtain the useful relations

$$\begin{aligned}&[\Gamma_i, \Gamma_j] = 0 \\ &A_i\Gamma_j = q\Gamma_j A_i \qquad A_i^\dagger \Gamma_j = q^{-1}\Gamma_j A_i^\dagger \qquad i \leqslant j \\ &[A_i, \Gamma_j] = [A_i^\dagger, \Gamma_j] = 0 \qquad i > j.\end{aligned} \tag{3.9}$$

As an illustration we take the last relation in (3.1):

$$\bar{A}_i\bar{A}_i^\dagger - q^{-2}\bar{A}_i^\dagger\bar{A}_i = 1 + (q^{-2}-1)\sum_{k=1}^{i-1}\bar{A}_k^\dagger\bar{A}_k. \tag{3.10}$$

Substituting (3.6) into (3.10) and multiplying $\Gamma_{i-1}\Gamma_i$ from both left and right, we obtain

$$A_iA_i^\dagger - A_i^\dagger A_i = \Gamma_{i-1}^2\Gamma_i^2\left\{1 - (q^2-1)\sum_{k=1}^{i-1}\Gamma_{k-1}^{-2}\Gamma_k^{-2}A_k^\dagger A_k\right\}. \tag{3.11}$$

Here the properties of Γ_i (3.10) were used. Because of the identity

$$\Gamma_i^2\left\{1 - (q^2-1)\sum_{k=1}^{i}\Gamma_{k-1}^{-2}\Gamma_k^{-2}A_k^\dagger A_k\right\} = 1 \tag{3.12}$$

(3.11) reads

$$A_i A_i^\dagger - A_i^\dagger A_i = \Gamma_i^2 = 1 + (q^2-1)\sum_{k=1}^{i} A_k^\dagger A_k.$$

Rearranging $A_i^\dagger A_i$, we obtain the last relation in (3.1).

The identity (3.12) is proved by mathematical induction. For $i = 1$, the left-hand side of (3.12) reads

$$\Gamma_1^2\{1-(q^2-1)\Gamma_1^{-2}A_1^\dagger A_1\} = \Gamma_1^2\Gamma_1^{-2}\{\Gamma_1^2 - (q^2-1)A_1^\dagger A_1\} = 1.$$

Assuming that (3.12) is valid for Γ_i, consider the case for Γ_{i+1}:

$$\begin{aligned}\Gamma_{i+1}^2\{1-(q^2-1)\sum_{k=1}^{i+1}\Gamma_{k-1}^{-2}\Gamma_k^{-2}A_k^\dagger A_k\} &= \Gamma_{i+1}^2\{\Gamma_i^{-2} - (q^2-1)\Gamma_i^{-2}\Gamma_{i+1}^{-2}A_{i+1}^\dagger A_{i+1}\}\\ &= \Gamma_i^{-2}\{\Gamma_{i+1}^2 - (q^2-1)A_{i+1}^\dagger A_{i+1}\}\\ &= \Gamma_i^{-2}\Gamma_i^2 = 1.\end{aligned}$$

The identity (3.12) has been proved.

From (3.6) it is obvious that φ is a one-to-one correspondence. In order to show that φ is an isomorphism, let us consider $\varphi' : \mathcal{A}_{q^{-1}} \to \mathcal{A}_q$ defined by

$$A_i = \bar\Gamma_{i-1}^{-1}\bar\Gamma_i^{-1}\bar A_i \qquad A_i^\dagger = \bar A_i^\dagger\bar\Gamma_{i-1}^{-1}\bar\Gamma_i^{-1}$$

and show that $\varphi\circ\varphi' = \varphi'\circ\varphi = 1$. To achieve this, it is enough to note the relation

$$\bar\Gamma_i = \sqrt{[\bar A_i, \bar A^\dagger{}_i]} = \sqrt{\Gamma_{i-1}^{-2}\Gamma_i^{-2}(A_iA_i^\dagger - q^2A_i^\dagger A_i)} = \Gamma_i^{-1}.$$

Therefore the statement has been proved. □

It is emphasized that, in the limit of $q \to 1$, both the left- and right-hand sides of (3.6) are reduced to the same bosonic oscillators.

4. q-deformed Lie algebra constructed from $\mathcal{A}_q$

As an application of the scheme provided in the previous section, let us consider the q-deformed Lie algebra which is constructed from $\mathcal{A}_q$. The bilinear forms of q-creation and q-annihilation operators can define a q-deformation of (the universal enveloping algebra of) the Lie algebra $u(n)$:

$$E_{ij} = A_i^\dagger A_j \qquad E_{ij}^* = E_{ji}. \tag{4.1}$$

Using (3.1) the commutation relations among the E_{ij}'s are obtained as

$$\begin{aligned}E_{ij}E_{kl} - q^{(\delta_{jk}-\delta_{il}+\theta(i-k)-\theta(j-l))}E_{kl}E_{ij} &= \delta_{jk}\left\{q^{-2(j-1)}E_{il} + (q^2-1)\sum_{m=1}^{j-1}q^{-2(j-m)}E_{im}E_{ml}\right\}\\ &\quad -\delta_{il}q^{\delta_{jk}-1+\theta(i-k)-\theta(j-l)}\left\{q^{-2(i-1)}E_{kj} + (q^2-1)\sum_{m=1}^{i-1}q^{-2(i-m)}E_{km}E_{mj}\right\}\end{aligned} \tag{4.2}$$

where $\theta(x)$ is a function defined by

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases} \tag{4.3}$$

The derivation of (4.2) is found in the appendix. In terms of the R-matrix, (4.2) is rewritten as

$$q \sum_{abcdef} R^{t_1}_{ab,\mu\nu} R^{-1}_{ap,cd} R_{cf,e\sigma} E_{ef} E_{db} - q^{-1} \sum_{abcdef} R_{\mu a,bc} R_{cd,e\nu} R^{t_1}_{fe,\rho\sigma} E_{fd} E_{ba}$$
$$= q\delta_{\rho\sigma} \sum_{ab} R^{t_1}_{ab,\mu\nu} E_{ab} - q^{-1}\delta_{\mu\nu} \sum_{ab} R^{t_1}_{ab,\rho\sigma} E_{ab} - q^{-1}\omega \sum_{abc} R^{t_1}_{ab,\mu\nu} R^{t_1}_{ca,\rho\sigma} E_{cb} \tag{4.4}$$

where t_1 means the transposition in the first space and $\omega \equiv q - q^{-1}$. The derivation of (4.4) is also sketched in the appendix. The $n = 2$ case of this algebra has been discussed in [11]. We can easily see that both (4.2) and (4.4) are reduced to the usual commutation relations of $u(n)$ in the limit of $q \to 1$:

$$[E_{\mu\nu}, E_{\rho\sigma}] = \delta_{\nu\rho} E_{\mu\sigma} - \delta_{\mu\sigma} E_{\rho\nu}$$

since the R-matrix is reduced to the unit matrix: $R_{ij,kl} \to \delta_{ik}\delta_{jl}$. As can be seen from (4.2) or (4.4), the obtained algebra is a quadratic deformation of $u(n)$ and differs from the standard Drinfeld–Jimbo deformation. A further difference from the standard deformation, as discussed in [11], is that the algebra describes the central extension of the q-deformation of $su(n)$ because the formula $u(n) = su(n) \oplus u(1)$ is not valid if $q \neq 1$.

We now adopt (4.2) or (4.4) as the definition of the non-standard q-deformed $u(n)$ and we do not use the covariant q-oscillator realization (4.1) hereafter. We shall investigate covariance and behaviour under $q \leftrightarrow q^{-1}$. The Hopf algebra structure for this q-deformed $u(n)$ is still an open problem.

The algebra (4.4) forms the $SU_q(n)$ tensor of rank (1,1), that is, the relation (4.4) is preserved by the transformation

$$E_{ij} \to E'_{ij} = \sum_{kl} t^*_{ik} t_{jl} E_{kl} \tag{4.5}$$

where we assume that E_{ij} commutes with t_{kl}. This can be proved by direct calculation using the properties of the R-matrix, i.e. (3.4), the Yang–Baxter equation and $R_{ij,kl} = R_{lk,ji}$.

We expect from (3.6) and (4.1) that the algebra (4.4) defined for q^{-1} is isomorphic to the algebra for q and the isomorphism is given by

$$\begin{aligned} \bar{E}_{ij} &= q^3 \Gamma^{-1}_{i-1} \Gamma^{-1}_{i} \Gamma^{-1}_{j-1} \Gamma^{-1}_{j} E_{ij} \qquad i < j \\ \bar{E}_{ii} &= q^2 \Gamma^{-2}_{i-1} \Gamma^{-2}_{i} E_{ii} \end{aligned} \tag{4.6}$$

where $\bar{E}_{ij}$ denotes the element of the q-deformed $u(n)$ defined for q^{-1}. However, it does not hold unless the covariant q-oscillator realization (4.1) is used except in the case of $n = 2$. In order to show this, let us first consider the case of $n = 2$. For $n = 2$, (4.2) is reduced to

$$\begin{aligned} &[E_{11}, E_{22}] = 0 \\ &E_{11}E_{12} - q^2 E_{12}E_{11} = E_{12} \qquad [E_{22}, E_{12}] = -E_{12} - (q^2-1)E_{12}E_{11} \\ &q^2 E_{12}E_{21} - E_{21}E_{12} = (q^2-1)E_{11}^2 + E_{11} - E_{22} \end{aligned} \tag{4.7}$$

and Γ_i $(i = 1, 2)$ are restricted to the second expression in (3.8):

$$\Gamma_1 = \sqrt{1 + (q^2-1)E_{11}} \qquad \Gamma_2 = \sqrt{1 + (q^2-1)(E_{11} + E_{22})}. \tag{4.8}$$

It is easy to see that $E_{11} + E_{22}$ is a central element of this algebra, therefore, Γ_2 is also a central element. The non-trivial commutation relations are given by

$$\Gamma_1 E_{12} = q E_{12} \Gamma_1 \qquad \Gamma_1 E_{21} = q^{-1} E_{21} \Gamma_1. \tag{4.9}$$

It can be proved by direct calculation that the isomorphism between two q-deformed $u(2)$ defined for q and q^{-1} is given by (4.6). Here, we give only one example, the last relation of (4.8):

$$q^{-2}\bar{E}_{12}\bar{E}_{21} - \bar{E}_{21}\bar{E}_{12} = (q^{-2}-1)\bar{E}_{11}^2 + \bar{E}_{11} - \bar{E}_{22}. \tag{4.10}$$

After substituting (4.6) into (4.10), we can arrange Γ_i to the left of E_{kl} by making use of (4.9):

$$\Gamma_1^{-2}\Gamma_2^{-2}(q^2E_{12}E_{21} - E_{12}E_{21}) = (1-q^2)\Gamma_1^{-2}E_{11}^2 + E_{11} - \Gamma_2^{-2}E_{22}$$

where we have dropped the common factor Γ_1^{-2}. Multiplying $\Gamma_1^2\Gamma_2^2$ from the left, we obtain the last equation of (4.8). Furthermore, because of (4.6),

$$\bar{\Gamma}_1 = \Gamma_1^{-1} \qquad \bar{\Gamma}_2 = \Gamma_2^{-1}$$

hold. Therefore, the isomorphism has been proved.

On the other hand, for $n \geqslant 3$, $\sum_{i=1}^n E_{ii}$ is no longer a central element so that the commutation relation between Γ_i and E_{kl} becomes quite complicated and the mechanism which is used in the proof for the $n = 2$ case, namely arranging Γ_i to the left of E_{kl}, does not work. Therefore, (4.6) does not give the expected isomorphism for $n \geqslant 3$.

5. $SU_q(n/m)$ covariant q-oscillator system

The isomorphisms discussed in section 3 can be generalized to the q-oscillator system covariant under the quantum super group $SU_q(n/m)$. The $SU_q(n/m)$ covariant q-oscillator system $\mathcal{B}_q$ is generated by $2n$ bosonic generators $\{A_i, A_i^\dagger,\ i = 1, 2, \ldots, n\}$ and $2m$ fermionic generators $\{B_r, B_r^\dagger,\ r = 1, 2, \ldots, m\}$. The *-anti-involution of a generator without dagger gives the corresponding one with dagger and *vice versa*. The $2(n+m)$ generators satisfy the defining relations [11]

$$\begin{aligned}
&A_iA_j = qA_jA_i \qquad i<j \\
&A_iA_j^\dagger = qA_j^\dagger A_i \qquad i \neq j \\
&A_iA_i^\dagger - q^2A_i^\dagger A_i = 1 + (q^2-1)\sum_{k=1}^{i-1} A_k^\dagger A_k \\
&A_iB_r = qB_rA_i \qquad A_iB_r^\dagger = qB_r^\dagger A_i \\
&B_rB_s = -qB_sB_r \qquad r<s \\
&B_rB_s^\dagger = -qB_s^\dagger B_r \qquad r \neq s \\
&B_rB_r^\dagger + B_r^\dagger B_r = 1 + (q^2-1)\sum_{k=1}^{n} A_k^\dagger A_k + (q^2-1)\sum_{s=1}^{r-1} B_s^\dagger B_s \\
&B_r^2 = (B_r^\dagger)^2 = 0
\end{aligned} \tag{5.1}$$

and their *-involution. The algebra $\mathcal{B}_q$ can form a rank one tensor of $SU_q(n/m)$ under the assumption that bosonic generators of $\mathcal{B}_q$ commute with all of $SU_q(n/m)$, fermionic generators of $\mathcal{B}_q$ commute with even generators of $SU_q(n/m)$, while they anticommute with odd generators of $SU_q(n/m)$. The co-action of $SU_q(n/m)$ on $\mathcal{B}_q$ is defined by

$$\alpha_i \to \alpha_i' = \sum_{j=1}^{n+m} t_{ij}\alpha_j$$

$$\alpha_i^\dagger \to \alpha_i^{\dagger\prime} = \sum_{j=1}^{n+m} (-)^{p(t_{ij})p(\alpha_j)} t_{ij}^* \alpha_j^\dagger \tag{5.2}$$

where $t_{ij} \in SU_q(n/m)$ and we have introduced the unified notation for $\mathcal{B}_q$:

$$\alpha_i = A_i \qquad \text{for} \qquad 1 \leqslant i \leqslant n$$
$$\alpha_{i+n} = B_i \qquad \text{for} \qquad 1 \leqslant i \leqslant m \quad \text{etc}$$

and $p(a)$ denotes the parity of operator a, namely $p(a) = 1$ for fermionic a or odd a, $p(a) = 0$ for bosonic a or even a.

The R-matrix for $SU_q(n/m)$ is given by [9]

$$R = \sum_i^{n+m} q^{1-2p(i)} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + w \sum_{i>j} e_{ij} \otimes e_{ji} \tag{5.3}$$

where $p(i)$ denotes the parity of ith basis vector. We prove the following relationship between $\mathcal{B}_q$ and $\mathcal{B}_{q^{-1}}$.

Proposition 5.1. There exists an isomorphism $\varphi : \mathcal{B}_q \to \mathcal{B}_{q^{-1}}$ such that φ transmutes the defining relations of $\mathcal{B}_{q^{-1}}$ into those of $\mathcal{B}_q$. The explicit formulae are given by

$$\bar{A}_i = \Gamma_{i-1}^{-1}\Gamma_i^{-1} A_i \qquad \bar{A}^\dagger{}_i = A_i^\dagger \Gamma_{i-1}^{-1}\Gamma_i^{-1}$$
$$\bar{B}_r = \Lambda_{r-1}^{-1}\Lambda_r^{-1} B_r \qquad \bar{B}^\dagger{}_r = B_r^\dagger \Lambda_{r-1}^{-1}\Lambda_r^{-1} \tag{5.4}$$

where

$$\Gamma_i \equiv \sqrt{[A_i, A_i^\dagger]} \qquad \Gamma_0 \equiv 1$$
$$\Lambda_r \equiv \sqrt{B_r B_r^\dagger + q^2 B_r^\dagger B_r} \qquad \Lambda_0 \equiv \Gamma_n. \tag{5.5}$$

Proof. As in the case of the $SU_q(n)$ covariant q-oscillator, the statement can be proved by direct calculations using the commutation relations among Γ_i, Λ_r and the generators of $\mathcal{B}_q$. Because of the last relation of (5.1), Λ_r $(r \neq 0)$ can be rewritten as

$$\Lambda_r = \left\{ 1 + (q^2-1)\sum_{k=1}^{n} A_k^\dagger A_k + (q^2-1)\sum_{s=1}^{r} B_s^\dagger B_s \right\}^{1/2}. \tag{5.6}$$

The *-anti-involution does not change Λ_r; $\Lambda_r^* = \Lambda_r$. Some useful relations can easily be shown:

$$[\Gamma_i, \Lambda_r] = [\Lambda_r, \Lambda_s] = 0$$
$$[B_r, \Gamma_i] = [B_r^\dagger, \Gamma_i] = 0$$
$$A_i\Lambda_r = q\Lambda_r A_i \qquad A_i^\dagger\Lambda_r = q^{-1}\Lambda_r A_i^\dagger \tag{5.7}$$
$$[B_r, \Lambda_s] = [B_r^\dagger, \Lambda_s] = 0 \qquad \text{for } s < r$$
$$B_r\Lambda_s = q\Lambda_s B_r \qquad B_r^\dagger\Lambda_s = q^{-1}\Lambda_s B_r^\dagger \qquad \text{for } s \geqslant r.$$

It is not difficult to prove the statement using these relations together with (3.10). As an example, we consider the last relation of (5.1). Again, we denote the generators of $\mathcal{B}_{q^{-1}}$ by the operators with a bar:

$$\bar{B}_r\bar{B}_r^\dagger + \bar{B}_r^\dagger\bar{B}_r = 1 + (q^{-2}-1)\sum_{k=1}^{n} \bar{A}_k^\dagger\bar{A}_k + (q^{-2}-1)\sum_{s=1}^{r-1} \bar{B}_s^\dagger\bar{B}_s. \tag{5.8}$$

We substitute (5.4) into (5.8), then using (3.9) and (5.7) we obtain

$$B_r B_r^\dagger + q^2 B_r^\dagger B_r = \Lambda_{r-1}^2 \Lambda_r^2 \left\{ 1 - (q^2-1)\sum_{k=1}^{n} \Gamma_{k-1}^{-2}\Gamma_k^{-2} A_k^\dagger A_k - (q^2-1)\sum_{s=1}^{r-1} \Lambda_{s-1}^{-2}\Lambda_s^{-2} B_s^\dagger B_s \right\}$$
$$= \Lambda_{r-1}^2 \Lambda_r^2 \left\{ \Gamma_n^{-2} - (q^2-1)\sum_{s=1}^{r-1} \Lambda_{s-1}^{-2}\Lambda_s^{-2} B_s^\dagger B_s \right\}. \tag{5.9}$$

Relation (3.12) was used to derive the last line. As is shown later, an analogous identity to (3.12) holds:

$$\Lambda_r^2 \left\{ \Gamma_n^{-2} - (q^2-1)\sum_{s=1}^{r} \Lambda_{s-1}^{-2}\Lambda_s^{-2} B_s^\dagger B_s \right\} = 1. \tag{5.10}$$

Because of this identity, (5.9) can be rewritten

$$B_r B_r^\dagger + q^2 B_r^\dagger B_r = \Lambda_r^2$$
$$= 1 + (q^2-1)\sum_{k=1}^{n} A_k^\dagger A_k + (q^2-1)\sum_{s=1}^{r} B_s^\dagger B_s.$$

Re-arranging $B_r^\dagger B_r$, we obtain the last relation of (5.1).

The identity (5.10) is proved by mathematical induction. For $r = 1$, the left-hand side of (5.10) reads

$$\Lambda_1^2\{\Gamma_n^{-2} - (q^2-1)\Lambda_1^{-2}\Lambda_0^{-2} B_1^\dagger B_1\} = \Gamma_n^{-2}\{\Lambda_1^2 - (q^2-1) B_1^\dagger B_1\}.$$

By definition of Λ_1, it is obviously reduced to unity. Assuming that (5.10) is valid for Λ_r, consider the case of Λ_{r+1}:

$$\Lambda_{r+1}^2 \left\{ \Gamma_n^{-2} - (q^2-1)\sum_{s=1}^{r+1} \Lambda_{s-1}^{-2}\Lambda_s^{-2} B_s^\dagger B_s \right\} = \Lambda_{r+1}^2\{\Lambda_r^{-2} - (q^2-1)\Lambda_r^{-2}\Lambda_{r+1}^{-2} B_{r+1}^\dagger B_{r+1}\}$$
$$= \Lambda_r^{-2}\{\Lambda_{r+1}^2 - (q^2-1) B_{r+1}^\dagger B_{r+1}\}$$
$$= 1.$$

Therefore, the identity (5.10) has been proved.

It can be easily seen that the map $\varphi' : \mathcal{B}_{q^{-1}} \to \mathcal{B}_q$ defined by

$$A_i = \bar{\Gamma}_{i-1}^{-1}\bar{\Gamma}_i^{-1}\bar{A}_i \qquad A_i^\dagger = \bar{A}_i^\dagger \bar{\Gamma}_{i-1}^{-1}\bar{\Gamma}_i^{-1}$$
$$B_r = \bar{\Lambda}_{r-1}^{-1}\bar{\Lambda}_r^{-1}\bar{B}_r \qquad B_r^\dagger = \bar{B}_r^\dagger \bar{\Lambda}_{r-1}^{-1}\bar{\Lambda}_r^{-1} \tag{5.11}$$

is the inverse map of φ, because of the relation

$$\bar{\Lambda}_r = \Lambda_r^{-1}. \tag{5.12}$$

This completes the proof of the statement. □

It is natural to apply the isomorphism to the q-deformation of the Lie superalgebra $u(n/m)$ constructed from $\mathcal{B}_q$. As in the case of the q-deformed $u(n)$ discussed in section 4, a trivial extension is not valid, against our expectation. Let us show this in the simplest case, namely q-deformed $u(1/1)$ [11]. The generators of q-deformed $u(1/1)$ are constructed by

$$Q = A^\dagger B \qquad Q^\dagger = B^\dagger A \qquad X = A^\dagger A \qquad Y = B^\dagger B. \tag{5.13}$$

They satisfy the commutation relations

$$Q^2 = 0 \qquad q^2 Q Q^\dagger + q^{-2} Q^\dagger Q = X + q^{-2} Y + (q^2-1) X^2$$
$$XQ - q^2 QX = Q \qquad YQ = 0 \qquad q^2 QY = Q + (q^2-1) XQ \qquad [X, Y] = 0 \tag{5.14}$$

and their *-involution. Again, we have obtained a quadratic deformation of a Lie algebra. We regard (5.14) as the defining relations of the q-deformed $u(1/1)$ and do not use the realization (5.13) by $\mathcal{B}_q$ hereafter. According to (5.13), we expect that the q-deformed $u(1/1)$ defined for q^{-1} is isomorphic to the one defined for q; the isomorphism is given by

$$\begin{aligned} &\bar{Q} = q^3\Gamma^{-2}\Lambda^{-1}Q \qquad \bar{Q}^\dagger = q^3 Q^\dagger \Gamma^{-2}\Lambda^{-1} \\ &\bar{X} = q^2\Gamma^{-2}X \qquad \bar{Y} = q^2\Gamma^{-2}\Lambda^{-2}Y \end{aligned} \tag{5.15}$$

where

$$\Gamma = \sqrt{1+(q^2-1)X} \qquad \Lambda = \sqrt{1+(q^2-1)(X+Y)}. \tag{5.16}$$

As an example, we consider the second equation of (5.14):

$$q^{-2}\bar{Q}\bar{Q}^\dagger + q^2\bar{Q}^\dagger\bar{Q} = \bar{X} + q^2\bar{Y} + (q^{-2}-1)\bar{X}^2.$$

Substituting (5.15) into this equation and multiplying $q^{-2}\Gamma^2\Lambda^2$ from the left, we obtain

$$q^2(QQ^\dagger + Q^\dagger Q) = X + (q^2-1)X^2 + q^2Y + (q^4-1)XY.$$

The correct equation cannot be derived unless the relation

$$q^2XY + Y - Q^\dagger Q = 0 \tag{5.17}$$

holds. However, (5.17) does not hold without the aid of the covariant q-oscillator realization (5.13). Therefore, we have shown that (5.15) does not give the isomorphism between two q-deformed $u(1/1)$ defined for q and q^{-1}.

6. Discussion

In this paper we have shown that, in the case of $SU_q(n)$ and $SU_q(n/m)$, the covariant q-oscillator systems defined for q are isomorphic to the ones for q^{-1}. The final goal of an investigation along the line presented here is to establish relationships between all kinds of q-deformed objects defined for q and q^{-1}. This is not easy, but is a challenging problem. As was seen in the case of q-deformed Lie algebras, the established isomorphism between covariant q-oscillator systems cannot be generalized directly to other q-deformed objects. For the q-deformed Lie algebras we will have to re-analyse the isomorphism based on the structure of the algebra itself without the aid of covariant q-oscillator realizations.

We can mention another example, an algebra generated by some copies of a covariant q-oscillator system. We require that the mutual commutation relations among various copies should also be covariant under the action of a certain quantum group. This requirement concludes that generators of a covariant q-oscillator system do not commute with their copies, and the commutation relations among various copies becomes non-trivial. For example, commutation relations between the $SU_q(n)$ covariant q-oscillator system $\mathcal{A}_q = \{A_i,\ A_i^\dagger\}$ and its copy $\{D_i, D_i^\dagger\}$ are given, in terms of the R-matrix, by [12]

$$\begin{aligned} &D_jA_i = q\sum_{kl} R_{ij,kl}A_kD_l \qquad A_j^\dagger D_i^\dagger = q\sum_{kl} R_{kl,ij}D_k^\dagger A_l^\dagger \\ &A_jD_i^\dagger = q\sum_{kl} R_{il,kj}D_k^\dagger A_l \qquad D_jA_i^\dagger = q\sum_{kl} R_{il,kj}A_k^\dagger D_l. \end{aligned} \tag{6.1}$$

It can be easily verified that (3.6) does not give the isomorphism between the algebra generated by $\{Ai,\ A_i^\dagger,\ D_i,\ D_i^\dagger\}$ and the algebra defined for q^{-1}, although we do not give the proof here. This is due to the additional structure given by (6.1) which we have to take into consideration if we wish to establish the isomorphism.

One of the most important problems concerning the isomorphism discussed here is the relationships between quantum groups defined for q and q^{-1}, e.g. $SU_q(n)$ and $SU_{q^{-1}}(n)$. Unfortunately, the result of this paper does not seem to be applicable to the problem; it will be a future work.

Appendix A.

The derivation of (4.2) and (4.4) is sketched in this appendix.

Let us first consider (4.2). It should be noted that the first and the second relations in (3.1) can be written using the function $\theta(x)$ defined in (4.3) as

$$A_i A_j = q^{-\theta(i-j)} A_j A_i \qquad A_i^\dagger A_j^\dagger = q^{\theta(i-j)} A_j^\dagger A_i^\dagger. \tag{A.1}$$

Another expression of the last relation of (3.1) is given by

$$A_i A_i^\dagger - q^2 A_i^\dagger A_i = q^{-2(i-1)} + (q^2-1) \sum_{m=1}^{i-1} q^{-2(i-m)} A_m A_m^\dagger.$$

This can be shown easily by making use of mathematical induction. Combining this with the third relation of (3.1), we obtain

$$A_i A_j^\dagger - q^{\delta_{ij}+1} A_j^\dagger A_i = \delta_{ij} \left\{ q^{-2(i-1)} + (q^2-1) \sum_{m=1}^{i-1} q^{-2(i-m)} A_m A_m^\dagger \right\}. \tag{A.2}$$

We can re-order $E_{ij}E_{kl} = A_i^\dagger A_j A_k^\dagger A_l$ as

$$A_i^\dagger A_j A_k^\dagger A_l \to A_i^\dagger A_k^\dagger A_j A_l \to A_k^\dagger A_i^\dagger A_l A_j \to A_k^\dagger A_l A_i^\dagger A_j \tag{A.3}$$

using (A.2) for the first and the third processes and (A.1) for the second process. This procedure gives us (4.2).

Next it should be noted that (3.1) can be expressed in terms of the R-matrix:

$$\begin{aligned} A_j A_i &= q^{-1} \sum_{kl} R_{ij,kl} A_k A_l \\ A_j^\dagger A_i^\dagger &= q^{-1} \sum_{kl} R_{kl,ij} A_k^\dagger A_l^\dagger \\ A_j A_i^\dagger &= \delta_{ij} + q \sum_{kl} R_{il,kj} A_k^\dagger A_l. \end{aligned} \tag{A.4}$$

Using (A.5), we again re-order $E_{ij}E_{kl}$ in the way of (A.3). After the second process is finished, we obtain

$$E_{ij}E_{kl} = \delta_{jk} E_{il} + q^{-1} \sum_{abcdst} R_{ka,bj} R_{cd,bi} R_{la,st} A_c^\dagger A_d^\dagger A_s A_t.$$

Multiplying by $R_{\rho l,\mu\sigma} R^{-1}_{\mu\nu,\alpha k} R_{\alpha j,i\beta}$ and summing over the indices i, j, k, l, α and μ, we obtain

$$\begin{aligned} &\sum_{ijkl\alpha\mu} R_{\rho l,\mu\sigma} R^{-1}_{\mu\nu,\alpha k} R_{\alpha j,i\beta} E_{ij} E_{kl} \\ &\quad = \sum_{l\mu} R_{\rho l,\mu\sigma} \delta_{\nu\beta} E_{\mu l} + q^{-1} \sum_{cdjlst\alpha\mu} R_{\rho t,\mu\alpha} R_{\mu s,dl} R_{\alpha l,j\sigma} R_{\nu j,c\beta} A_c^\dagger A_d^\dagger A_s A_t \\ &\quad = \sum_{l\mu} R_{\rho l,\mu\sigma} \delta_{\nu\beta} E_{\mu l} - q^{-2} \sum_{cjlt\alpha} R_{\rho t,l\alpha} R_{\alpha l,j\sigma} R_{\nu j,c\beta} E_{ct} \\ &\qquad + q^{-2} \sum_{cjlt\alpha\mu} R_{\rho t,\mu\alpha} R_{\alpha l,j\sigma} R_{\nu j,c\beta} E_{cl} E_{\mu t}. \end{aligned} \tag{A.5}$$

The second equality corresponds to the third process of (A.3). Applying the Hecke condition

$$\check{R}^2 = \omega \check{R} + 1 \qquad \check{R}_{ij,kl} = R_{ji,kl}$$

to the second term of (A.5), we obtain (4.4).

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